Nonabelian Hodge theory and the decomposition theorem for 2-CY categories

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The BBDG decomposition theorem

Theorem (Beilinson–Bernstein–Deligne–Gabber)

Let $p\colon X \to Y$ be a projective morphism of complex algebraic varieties, with X smooth. Then

$$p_*\mathbb{Q}_X \cong \bigoplus_{i\in\mathbb{Z}} {}^{\mathfrak{p}}\mathcal{H}^i(p_*\mathbb{Q}_X)\,[-i]$$

and each perverse sheaf ${}^{\mathfrak{p}}\mathcal{H}^i(p_*\mathbb{Q}_X)$ is semisimple.

- If X is singular, same theorem remains true, but with $\mathbb{Q}_X$ replaced everywhere by $\mathcal{IC}_X(\mathbb{Q})$, the intersection complex, i.e. the intermediate extension of the constant perverse sheaf $\mathbb{Q}[\dim(X)]|_{X^{\mathrm{sm}}}$.
- For X smooth, defining

$$\mathfrak{P}^i\mathrm{H}(X,\mathbb{Q}) := \mathrm{H}(Y, {}^{\mathfrak{p}}\tau^{\leq i}p_*\mathbb{Q}_X) \subset \mathrm{H}(X,\mathbb{Q})$$

gives the perverse filtration of X with respect to p, with subquotients

$$\mathfrak{P}^i\mathrm{H}(X,\mathbb{Q})/\mathfrak{P}^{i-1}\mathrm{H}(X,\mathbb{Q}) \cong \mathrm{H}(Y, {}^{\mathfrak{p}}\mathcal{H}^i(p_*\mathbb{Q}_X)).$$

Saito's version

Given X a complex variety, Saito defines the category $\mathrm{MHM}(X)$ of mixed Hodge modules on X, along with faithful functor $\mathrm{MHM}(X) \to \mathrm{Perv}(X)$;

- So a MHM on X is a perverse sheaf $\mathcal{F}$ on X along with some extra data. Most important part (for us) is the *weight filtration* $W_\bullet \mathcal{F}$.
- A MHM $\mathcal{F}$ is *pure of weight* i if $\mathrm{Gr}_j^W \mathcal{F} = 0$ for $i \neq j$.
- A complex $\mathcal{F} \in D^b(\mathrm{MHM}(X))$ is called *pure* if each $\mathcal{H}^i(\mathcal{F})$ is pure of weight i.
- The perverse sheaves $\mathbb{Q}_X[\dim(X)]$ on a smooth variety X (or $\mathcal{IC}_X(\mathbb{Q})$ on a general variety) have lifts to simple pure weight zero MHMs (at least if $\dim(X)$ even).

Theorem (Saito)

The category of pure weight n MHMs on a variety is semisimple. If $f\colon X \to Y$ is projective, $f_*\colon D^b(\mathrm{MHM}(X)) \to D^b(\mathrm{MHM}(Y))$ preserves pure objects. If $\mathcal{F} \in D^b(\mathrm{MHM}(Y))$ is pure, then $\mathcal{F} \cong \bigoplus_{i\in\mathbb{Z}} \mathcal{H}^i(\mathcal{F})[-i]$. $\therefore$ If $\mathcal{G} \in D^b(\mathrm{MHM}(X))$ is pure, then $f_*\mathcal{G} \cong \bigoplus_{i\in\mathbb{Z}} \mathcal{H}^i(f_*\mathcal{G})[-i]$.

Higgs bundles

Throughout, C will denote a smooth projective complex curve of genus g.

Definition

A Higgs bundle on C is a pair $\tilde{\mathcal{F}} = (\mathcal{F}, \eta)$, where $\mathcal{F}$ is a locally free coherent sheaf on C, and $\eta \colon \mathcal{F} \to \mathcal{F} \otimes \omega_C$ is the *Higgs field*. The rank and degree of $\tilde{\mathcal{F}}$ is defined to be the rank and degree of $\mathcal{F}$.

- A morphism $\tilde{h} \colon \tilde{\mathcal{G}} \to \tilde{\mathcal{F}}$ is a morphism $h \colon \mathcal{G} \to \mathcal{F}$ commuting with the Higgs fields: $\eta_{\mathcal{F}} \circ h = (h \otimes \omega_C) \circ \eta_{\mathcal{G}}$.
- We define the *slope* $\mu(\tilde{\mathcal{F}}) = \deg(\mathcal{F})/\mathrm{rank}(\mathcal{F})$. We call $\tilde{\mathcal{F}}$ *semistable* if $\mu(\tilde{\mathcal{G}}) \leq \mu(\tilde{\mathcal{F}})$ for all nonzero $\tilde{\mathcal{G}} \subsetneq \tilde{\mathcal{F}}$, and *stable* if the inequalities are strict.

Categorical structure

The category $\mathrm{Higgs}(C)$ is a *2-Calabi–Yau category*. In particular, there are bifunctorial isomorphisms $\mathrm{Ext}^i(\tilde{\mathcal{F}}, \tilde{\mathcal{G}}) \cong \mathrm{Ext}^{2-i}(\tilde{\mathcal{G}}, \tilde{\mathcal{F}})^\vee$.

Dolbeault moduli spaces

- Fix a curve C, and numbers r, d. We define $\mathfrak{M}^{\mathrm{Dol}}_{r,d}(C)$ to be the moduli stack of semistable rank r, degree d Higgs bundles on C, and $\mathcal{M}^{\mathrm{Dol}}_{r,d}(C)$ the coarse moduli space. Points of $\mathcal{M}^{\mathrm{Dol}}_{r,d}(C)$ correspond to polystable Higgs bundles.
- There is a canonical morphism $p\colon \mathfrak{M}^{\mathrm{Dol}}_{r,d}(C) \to \mathcal{M}^{\mathrm{Dol}}_{r,d}(C)$ sending a semistable Higgs bundle to associated polystable Higgs bundle.
- If $(r, d) = 1$ then $\mathcal{M}^{\mathrm{Dol}}_{r,d}(C)$ is smooth, and $\mathfrak{M}^{\mathrm{Dol}}_{r,d}(C)$ is a $\mathbb{C}^*$ gerbe over it (i.e. fibres of p are $\mathrm{B}\mathbb{C}^*$).
- We consider the *Hitchin map* $\mathtt{h}\colon \mathcal{M}^{\mathrm{Dol}}_{r,d}(C) \to \Lambda_r := \prod_{i=1}^{r} \mathrm{H}^0(C, \omega_C^{\otimes i})$ recording the eigenvalues of the Higgs field:
  $$\mathtt{h}((\mathcal{F}, \eta)) = (\mathrm{Tr}(\eta), \mathrm{Tr}(\eta^2), \ldots, \mathrm{Tr}(\eta^r)).$$
- The morphism $\mathtt{h}$ is projective, and so for all r, d, the complex $\mathtt{h}_* \mathcal{IC}_{\mathcal{M}^{\mathrm{Dol}}_{r,d}(C)}(\mathbb{Q})$ splits, and the intersection cohomology $\mathrm{IC}(\mathcal{M}^{\mathrm{Dol}}_{r,d}(C))$ acquires a perverse filtration defined with respect to $\mathtt{h}$:

$$\mathfrak{P}^i_{\mathtt{h}} \mathrm{IC}(\mathcal{M}^{\mathrm{Dol}}_{r,d}(C)) := \mathrm{H}(\Lambda_r, {}^{\mathfrak{p}}\tau^{\leq i}\, \mathtt{h}_* \mathcal{IC}_{\mathcal{M}^{\mathrm{Dol}}_{r,d}(C)}(\mathbb{Q})).$$

The Betti side

Let Σ_g be a genus g compact Riemann surface, e.g. the underlying topological space of C_{an}. Let $\Sigma'_g = \Sigma_g \setminus \{c\}$ for some $c \in \Sigma_g$. Then we have the standard presentations

$$\begin{aligned}\pi_1(\Sigma_g) =& \langle a_1, \ldots, a_g, b_1, \ldots, b_g | \prod_{i=1}^{g} (a_i, b_i) \rangle \\ \pi_1(\Sigma'_g) =& \langle a_1, \ldots, a_g, b_1, \ldots, b_g \rangle\end{aligned}$$

- Let $\mathfrak{M}^{\mathrm{B}}_{g,r,d}$ be the stack of r-dimensional $\pi_1(\Sigma'_g)$-reps for which the action of $\prod_{i=1}^{g}(a_i, b_i)$ is multiplication by $\exp(2\pi id/r)$.
- So $\mathfrak{M}^{\mathrm{B}}_{g,r,0}$ is $\mathfrak{M}_r(\pi_1(\Sigma_g))$, the stack of r-dimensional $\mathbb{C}[\pi_1(\Sigma_g)]$-modules.
- Let $\mathcal{M}^{\mathrm{B}}_{g,r,d}$ be the coarse moduli space: points correspond to semisimple $\pi_1(\Sigma'_g)$-modules such that $\prod_{i=1}^{g}(a_i, b_i)$ acts via multiplication by $\exp(2\pi id/r)$.
- If $(r, d) = 1$ then $\mathcal{M}^{\mathrm{B}}_{g,r,d}$ is a smooth variety, and $\mathfrak{M}^{\mathrm{B}}_{g,r,d}$ is a $\mathbb{C}^*$-gerbe over it.

Nonabelian Hodge theory

Theorem (Hitchin, Donaldson, Corlette, Simpson)

Fix C a smooth projective genus g curve. For all r, d there is a homeomorphism

$$\Psi\colon \mathcal{M}^{\mathrm{B}}_{g,r,d} \xrightarrow{\cong} \mathcal{M}^{\mathrm{Dol}}_{r,d}(C).$$

$\therefore$ *there is an isomorphism* $\mathrm{H}(\Psi)\colon \mathrm{H}(\mathcal{M}^{\mathrm{B}}_{g,r,d}, \mathbb{Q}) \xrightarrow{\cong} \mathrm{H}(\mathcal{M}^{\mathrm{Dol}}_{r,d}(C), \mathbb{Q})$

- If $(r, d) = 1$ there is an isomorphism $\mathrm{H}(\mathfrak{M}^{\mathrm{B}}_{g,r,d}, \mathbb{Q}) \xrightarrow{\cong} \mathrm{H}(\mathfrak{M}^{\mathrm{Dol}}_{r,d}(C), \mathbb{Q})$ since $\mathrm{H}(\mathfrak{M}^{\mathrm{B}}_{g,r,d}, \mathbb{Q}) \cong \mathrm{H}(\mathcal{M}^{\mathrm{B}}_{g,r,d}, \mathbb{Q}) \otimes \mathrm{H}(\mathrm{B}\,\mathbb{C}^*, \mathbb{Q})$ and similarly for Dolbeault side.
- For all r, d there is an isomorphism $\mathrm{IC}(\mathcal{M}^{\mathrm{B}}_{g,r,d}) \cong \mathrm{IC}(\mathcal{M}^{\mathrm{Dol}}_{r,d}(C))$ since intersection complex is a topological invariant.

Conjecture (P=W conjecture)

For r, d coprime, $\mathrm{H}(\Psi)\left(W_{2i}\,\mathrm{H}(\mathcal{M}^{\mathrm{B}}_{g,r,d}, \mathbb{Q}_{\mathrm{vir}})\right) = \mathfrak{P}^{i}_{\mathfrak{h}}\,\mathrm{H}(\mathcal{M}^{\mathrm{Dol}}_{r,d}(C), \mathbb{Q}_{\mathrm{vir}})$.

NAHT for stacks...?

For $\mathfrak{M}$ a singular stack of objects in a 2CY category we define

$$\mathrm{H}^{\mathrm{BM}}(\mathfrak{M},\mathbb{Q})_{\mathrm{vir}} := \mathrm{H}\left(\mathfrak{M},\mathbb{D}\mathbb{Q}_{\mathfrak{M}}\otimes\mathbb{L}^{-\chi(\cdot,\cdot)/2}\right)\cong \mathrm{H}_c(\mathfrak{M},\mathbb{Q}))^{\vee}\otimes\mathbb{L}^{-\chi(\cdot,\cdot)/2}$$

Define the stacky Hitchin map

$$\mathtt{h}_{\mathrm{St}}\colon \mathfrak{M}^{\mathrm{Dol}}_{r,d}(C)\rightarrow \Lambda_r$$
$$(\mathcal{F},\eta)\mapsto (\mathrm{Tr}(\eta),\mathrm{Tr}(\eta^2),\ldots,\mathrm{Tr}(\eta^r))$$

"Conjecture"

- $\exists$ nat. iso. $\mathrm{H}^{\mathrm{BM}}(\Psi)\colon\ \mathrm{H}^{\mathrm{BM}}(\mathfrak{M}^{\mathrm{B}}_{g,r,d},\mathbb{Q})_{\mathrm{vir}}\xrightarrow{\cong}\mathrm{H}^{\mathrm{BM}}(\mathfrak{M}^{\mathrm{Dol}}_{r,d}(C),\mathbb{Q})_{\mathrm{vir}}$
- $\mathrm{H}^{\mathrm{BM}}(\Psi)$ sends $W_{2i}(\mathrm{H}^{\mathrm{BM}}(\mathfrak{M}^{\mathrm{B}}_{g,r,d},\mathbb{Q})_{\mathrm{vir}})$ to $\mathfrak{P}^{i}_{\mathtt{h}_{\mathrm{St}}}\mathrm{H}^{\mathrm{BM}}(\mathfrak{M}^{\mathrm{Dol}}_{r,d}(C),\mathbb{Q})_{\mathrm{vir}}$

Three problems with the conjecture:

1. For perverse filtration, we have assumed that $\mathtt{h}_{\mathrm{St},*}\,\mathbb{D}\mathbb{Q}_{\mathfrak{M}^{\mathrm{Dol}}_{r,d}(C),\mathrm{vir}}$ splits.
2. We still need to construct $\mathrm{H}^{\mathrm{BM}}(\Psi)$.
3. Conjecture is false!!

The case $g = 0$

1. Let $\tilde{\mathcal{F}} = (\mathcal{F}, \eta)$ be a semistable rank r degree zero Higgs bundle on $\mathbb{P}^1$. Then $\mathcal{F} \cong \mathcal{O}_{\mathbb{P}^1}^{\oplus r}$ and $\eta = 0$.

$$\begin{array}{ccc} \mathfrak{M}^{\mathrm{Dol}}_{r,0}(C) & \xrightarrow{\cong} & \mathrm{pt}\,/\,\mathrm{GL}_r \\ \downarrow{\scriptstyle \mathtt{h}} & & \downarrow \\ \Lambda_r & \xrightarrow{\cong} & \mathrm{pt}\,. \end{array}$$

   Since $\mathtt{h}$ is the structure morphism, $\mathfrak{P}^{\bullet}_{\mathtt{h}}\,\mathrm{H}(\mathfrak{M}^{\mathrm{Dol}}_{r,0}(C), \mathbb{Q})$ is the filtration by cohomological degree.

2. $\pi_1(\mathbb{P}_1) = \langle 1 \rangle$ and so likewise $\mathfrak{M}^{\mathrm{B}}_{g,r,0} \cong \mathrm{pt}\,/\,\mathrm{GL}_r$. The MHS $\mathrm{H}(\mathrm{pt}\,/\,\mathrm{GL}_r, \mathbb{Q})$ is pure (Deligne), so the weight filtration is the filtration by cohomological degree.

$\therefore$ There is an isomorphism of stacks $\Psi\colon \mathfrak{M}^{\mathrm{B}}_{g,r,0} \cong \mathfrak{M}^{\mathrm{Dol}}_{r,0}(C)$. But: no halving of weight degrees!

$\mathrm{H}(\Psi)\left(W_i(\mathrm{H}(\mathfrak{M}^{\mathrm{B}}_{g,r,0}, \underline{\mathbb{Q}}))\right) = \mathfrak{P}^{i}_{\mathtt{h}_{\mathrm{St}}}\mathrm{H}(\mathfrak{M}^{\mathrm{Dol}}_{r,0}(C), \mathbb{Q}).$

2CY categories

Examples of 2CY categories

1. Higgs bundles on a smooth projective curve
2. $\pi_1(\Sigma_g)$-representations
3. Categories of modules over (ordinary/deformed/multiplicative) preprojective algebras
4. Compactly supported coherent sheaves on a smooth surface S satisfying $\omega_S \cong \mathcal{O}_S$
5. Kuznetsov components.

- We restrict attention to stacks of objects $\mathfrak{M}_{\mathcal{C}}$ possessing *good moduli spaces* $p\colon \mathfrak{M}_{\mathcal{C}} \to \mathcal{M}_{\mathcal{C}}$ in the sense of Alper.
- This means that in cases (1), (4), (5) we restrict attention to semistable objects.
- In all cases apart from (5), the stacks $\mathfrak{M}_{\mathcal{C}}$ have known global quotient presentations.

The decomposition theorem for 2CY categories

Theorem (2CY decomposition theorem)

- *Let $\mathfrak{M}_{\mathcal{C}}$ be an open substack of the stack of objects in a 2CY category $\mathcal{C}$, and assume that there is a good moduli space $p\colon \mathfrak{M}_{\mathcal{C}} \to \mathcal{M}_{\mathcal{C}}$. Then $\mathcal{H}^i(p_*\mathbb{D}\underline{\mathbb{Q}}_{\mathfrak{M}_{\mathcal{C}},\mathrm{vir}})$ is pure of weight i, and vanishes for $i \notin 2\mathbb{Z}_{\geq 0}$.*

Corollary

For $\mathcal{C}$ a 2CY category from cases (1)-(4), the (pure) complex $p_*\mathbb{D}\underline{\mathbb{Q}}_{\mathfrak{M}_{\mathcal{C}},\mathrm{vir}}$ splits, yielding the perverse filtration $\mathfrak{P}^{\bullet}_{p}\,\mathrm{H}^{\mathrm{BM}}(\mathfrak{M}_{\mathcal{C}}, \underline{\mathbb{Q}}_{\mathrm{vir}})$.

Since $\mathtt{h}_{\mathrm{St},*}\,\mathbb{D}\underline{\mathbb{Q}}_{\mathfrak{M}^{\mathrm{Dol}}_{r,d}(C),\mathrm{vir}} \cong \mathtt{h}_*\,p_*\mathbb{D}\underline{\mathbb{Q}}_{\mathfrak{M}^{\mathrm{Dol}}_{r,d}(C),\mathrm{vir}}$ and $\mathtt{h}$ is projective, we deduce

Corollary

The complex $\mathtt{h}_{\mathrm{St},*}\mathbb{D}\underline{\mathbb{Q}}_{\mathfrak{M}^{\mathrm{Dol}}_{r,d}(C),\mathrm{vir}}$ is pure, yielding the perverse filtration $\mathfrak{P}^{\bullet}_{\mathtt{h}_{\mathrm{St}}}\,\mathrm{H}^{\mathrm{BM}}(\mathfrak{M}^{\mathrm{Dol}}_{r,d}(C), \underline{\mathbb{Q}}_{\mathrm{vir}})$. Problem 1 solved!

Solving problem 3: the "correct" SP=SW conjecture(s)

For p the morphism from the stack to the GIT quotient, there are filtrations

$$\mathfrak{P}_p^{\bullet}\,\mathrm{H}^{\mathrm{BM}}(\mathfrak{M}^{\mathrm{B}}_{g,r,0},\mathbb{Q}_{\mathrm{vir}}) \qquad \mathfrak{P}_p^{\bullet}\,\mathrm{H}^{\mathrm{BM}}(\mathfrak{M}^{\mathrm{Dol}}_{r,0}(C),\mathbb{Q}_{\mathrm{vir}}).$$

Conjecture (SP=SW version 1)

$\exists$ isomorphism $\mathfrak{P}_p^{0}\,\mathrm{H}^{\mathrm{BM}}(\mathfrak{M}^{\mathrm{B}}_{g,r,0},\mathbb{Q}_{\mathrm{vir}}) \xrightarrow{\cong} \mathfrak{P}_p^{0}\,\mathrm{H}^{\mathrm{BM}}(\mathfrak{M}^{\mathrm{Dol}}_{r,0}(C),\mathbb{Q}_{\mathrm{vir}})$ *sending* $W_{2i}\mathfrak{P}_p^{0}\,\mathrm{H}^{\mathrm{BM}}(\mathfrak{M}^{\mathrm{B}}_{g,r,0},\mathbb{Q}_{\mathrm{vir}})$ *to* $\mathfrak{P}^{i}_{\mathtt{h}_{\mathrm{St}}}\mathfrak{P}_p^{0}\,\mathrm{H}^{\mathrm{BM}}(\mathfrak{M}^{\mathrm{Dol}}_{r,0}(C),\mathbb{Q}_{\mathrm{vir}})$

Conjecture (SP=SW version 2)

$\exists$ isomorphism $\mathrm{H}^{\mathrm{BM}}(\mathfrak{M}^{\mathrm{B}}_{g,r,0},\mathbb{Q}_{\mathrm{vir}}) \xrightarrow{\cong} \mathrm{H}^{\mathrm{BM}}(\mathfrak{M}^{\mathrm{Dol}}_{r,0}(C),\mathbb{Q}_{\mathrm{vir}})$ *sending* $W_{2i}\mathrm{H}^{\mathrm{BM}}(\mathfrak{M}^{\mathrm{B}}_{g,r,0},\mathbb{Q}_{\mathrm{vir}})$ *to* $\sum_{j+k=i}\mathfrak{P}^{j+2k}_{\mathtt{h}_{\mathrm{St}}}\mathfrak{P}_p^{2k}\mathrm{H}^{\mathrm{BM}}(\mathfrak{M}^{\mathrm{Dol}}_{r,0}(C),\mathbb{Q}_{\mathrm{vir}})$

Theorem

The SP=SW conjectures are true for $g \leq 1$.

Étale neighbourhoods

Formality

For $\mathcal{F}_1, \ldots, \mathcal{F}_m$ a collection of simple objects in a 2CY category $\mathcal{C}$, the A_∞-Yoneda algebra $A = \mathrm{Ext}_\infty(\bigoplus_i \mathcal{F}_i, \bigoplus_i \mathcal{F}_i)$ is *formal*, i.e. up to isomorphism of A_∞ algebras, it is an ordinary associative algebra. It follows that A is determined, up to iso, by the dimensions of the $\mathrm{Ext}^1(\mathcal{F}_i, \mathcal{F}_j)$.

Modular étale neighbourhoods

Let $\mathcal{F}_1, \ldots, \mathcal{F}_m$ be as above. Let $x \in \mathfrak{M}_{\mathcal{C}}$ represent the object $\bigoplus_i \mathcal{F}_i^{\oplus \mathrm{d}_i}$. Define Q with vertices $\{\mathcal{F}_1, \ldots, \mathcal{F}_m\}$ such that $\overline{Q}$ is the Ext quiver of the $\mathcal{F}_i$. Then there is a commutative diagram

$$\begin{array}{ccccc}
(\mathfrak{M}_{\mathrm{d}}(\Pi_Q), 0_{\mathrm{d}}) & \hookleftarrow & (\mathfrak{U}, u) & \hookrightarrow & (\mathfrak{M}_{\mathcal{C}}, x) \\
\downarrow{\scriptstyle \overline{\mathrm{JH}}} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\
(\mathcal{M}_{\mathrm{d}}(\Pi_Q), 0_{\mathrm{d}}) & \hookleftarrow & (\mathcal{U}, q(u)) & \hookrightarrow & (\mathcal{M}_{\mathcal{C}}, x)
\end{array}$$

with Cartesian squares, and étale horizontal maps.

Cohomological integrality

- Let $p\colon \mathfrak{M}_{\mathcal{C}} \to \mathcal{M}_{\mathcal{C}}$ be the morphism from the stack of objects in a 2CY category to its coarse moduli space. Let $s\colon \mathcal{M}_{\mathcal{C}}^{\times 2} \to \mathcal{M}_{\mathcal{C}}$ be the morphism taking a pair of polystable objects to their direct sum.
- Then $D^b(\mathrm{MHM}(\mathcal{M}_{\mathcal{C}}))$ carries a *convolution product*

$$\mathcal{F} \boxtimes_{\oplus} \mathcal{G} := s_*(\mathcal{F} \boxtimes \mathcal{G})$$

- We say that *cohomological integrality* holds for $\mathcal{C}$ if we can write

$$p_* \mathbb{D}\mathbb{Q}_{\mathfrak{M}_{\mathcal{C}},\mathrm{vir}} \cong \mathrm{Sym}_{\boxtimes_{\oplus}}\left(\mathcal{BPS}_{\mathcal{C}} \otimes \mathrm{H}(\mathrm{B}\mathbb{C}^*, \mathbb{Q})\right)$$

  for some $\mathcal{BPS}_{\mathcal{C}} \in \mathrm{MHM}(\mathcal{M}_{\mathcal{C}})$.
- Cohomological integrality implies

$$\mathrm{H}^{\mathrm{BM}}(\mathfrak{M}_{\mathcal{C}}, \mathbb{Q}_{\mathrm{vir}}) \cong \mathrm{Sym}(\mathrm{BPS}_{\mathcal{C}} \otimes \mathrm{H}(\mathrm{B}\mathbb{C}^*, \mathbb{Q}))$$

  with

$$\mathrm{BPS}_{\mathcal{C}} := \mathrm{H}(\mathcal{M}_{\mathcal{C}}, \mathcal{BPS}_{\mathcal{C}}).$$

BPS cohomology

Theorem

Cohomological integrality holds for Π_Q-mod*. Moreover the BPS sheaves* $\mathcal{BPS}_{\Pi_Q\text{-mod}}$ *are pure, and the zeroth BPS cohomology* $\mathrm{H}^0(\mathcal{M}_{\mathcal{C}}, \mathcal{BPS}_{\Pi_Q\text{-mod}})$ *is the Kac–Moody Lie algebra associated to* Q*.*

Combining with the modular étale nbhd theorem, purity of BPS sheaves implies the 2CY decomposition theorem.

Theorem

Cohomological integrality holds for $\mathcal{C} = \mathbb{C}[\pi_1(\Sigma_g)]$-mod*. Moreover the BPS cohomology satisfies*

$$\chi_{\mathrm{wt}}(\mathrm{H}(\mathcal{M}_{\mathcal{C}}, \mathcal{BPS}_{\mathcal{C},r})) = \chi_{\mathrm{wt}}(\mathrm{H}(\mathcal{M}^{\mathrm{B}}_{g,r,1}(C), \mathbb{Q}_{\mathrm{vir}}))$$

Theorem (Kinjo,Koseki)

Let $\mathcal{C}$ *be the category of semistable degree zero Higgs bundles. Then cohomological integrality holds for* $\mathcal{C}$*, and* $\mathtt{h}_* \mathcal{BPS}_{\mathcal{C}} \cong \mathtt{h}_* \mathbb{Q}_{\mathcal{M}^{\mathrm{Dol}}_{r,1}(C),\mathrm{vir}}$*.*

Cohomological Hall algebras

We can equip $p_*\mathbb{D}\mathbb{Q}_{\mathfrak{M}_{\mathcal{C}},\mathrm{vir}}$ with a Hall algebra structure (for the tensor structure $\boxtimes_\oplus$), and the cohomological integrality isomorphism

$$p_*\mathbb{D}\mathbb{Q}_{\mathfrak{M}_{\mathcal{C}},\mathrm{vir}} \cong \mathrm{Sym}_{\boxtimes_\oplus}(\mathcal{BPS}_{\mathcal{C}} \otimes \mathrm{H}(\mathrm{B}\mathbb{C}^*, \mathbb{Q}))$$

becomes a PBW isomorphism. Applying $\tau^{\leq 0}$, we get the PBW isomorphism

$$\tau^{\leq 0} p_*\mathbb{D}\mathbb{Q}_{\mathfrak{M}_{\mathcal{C}},\mathrm{vir}} \cong \mathrm{U}_{\mathcal{C}} := \mathrm{Sym}_{\boxtimes_\oplus}(\mathcal{BPS}_{\mathcal{C}})$$

of algebra objects in $\mathrm{MHM}(\mathcal{M}_{\mathcal{C}})$. Since $\mathrm{U}_{\mathcal{C}}$ is pure, it is semisimple.

Trichotomy of generators

Let ρ be a simple object in $\mathcal{C}$. Either $\dim(\mathrm{Ext}^1(\mathcal{F},\mathcal{F}))$

- $=0$: then $\{\rho\}$ is a component of $\mathcal{M}_{\mathcal{C}}$ and $\mathbb{Q}_{\{\rho\}}$ is a summand of $\mathrm{U}_{\mathcal{C}}$,
- $=2$: then $\Delta_n \mathcal{IC}_E(\mathbb{Q})$ is a summand of $\mathrm{U}_{\mathcal{C}}$ for all $n \geq 1$, where E is the 2-dimensional component of $\mathcal{M}_{\mathcal{C}}$ containing ρ.
- >2: then $\mathcal{IC}_E(\mathbb{Q})$ is a summand of $\mathrm{U}_{\mathcal{C}}$, where $E \subset \mathcal{M}_{\mathcal{C}}$ is the connected component containing ρ.

In each case these summands are primitive, i.e. generators of $\mathrm{U}_{\mathcal{C}}$.

The NAHT isomorphism for BPS cohomology

- I conjecture that these are *all* the generators, and the algebra they generate is a Borcherds algebra, with the trichotomy corresponding to real, imaginary isotropic, and imaginary hyperbolic roots.
- For Higgs bundles, the trichotomy corresponds to $\mathrm{g} = 0, 1, \geq 2$. So for $\mathcal{H}$ the category of semistable degree zero Higgs bundles on a curve with $g(C) \geq 2$, all roots expected to be hyperbolic: giving isomorphism in $\mathrm{MHM}(\mathcal{M}_{\mathcal{H}})$

$$\mathcal{BPS}_{\mathcal{H}} \cong \mathrm{Free}_{\mathrm{Lie}}\left(\mathcal{IC}_{\mathcal{M}_{\mathcal{H}}}(\mathbb{Q})\right)$$

- Similarly for $\mathcal{B} = \mathbb{C}[\pi_1(\Sigma_g)]$-mod with $g \geq 2$

$$\mathcal{BPS}_{\mathcal{B}} \cong \mathrm{Free}_{\mathrm{Lie}}\left(\mathcal{IC}_{\mathcal{M}_{\mathcal{B}}}(\mathbb{Q})\right)$$

- Intersection cohomology is a topological invariant, and NAHT correspondence gives diffeo $\mathcal{M}_{\mathcal{H}} \cong \mathcal{M}_{\mathcal{B}}$, so we expect isomorphisms of BPS cohomology

$$\mathrm{BPS}_{\mathcal{H}} \cong \mathrm{Free}_{\mathrm{Lie}}(\mathrm{IC}(\mathcal{M}_{\mathcal{H}})) \cong \mathrm{Free}_{\mathrm{Lie}}(\mathrm{IC}(\mathcal{M}_{\mathcal{B}})) \cong \mathrm{BPS}_{\mathcal{B}}$$

The NAHT isomorphism for stacks

We put everything together to produce a NAHT iso. for BM homology of stacks. As on previous slide $\mathcal{H}$ is category of semistable degree zero Higgs bundles on genus g curve and $\mathcal{B} = \mathbb{C}[\pi_1(\Sigma_g)]$-mod with $g \geq 2$.

- From previous slide we expect a canonical isomorphism of Borcherds algebras

$$\mathsf{BPS}_{\mathcal{H}} \cong \mathsf{BPS}_{\mathcal{B}}$$

- By cohomological integrality there are canonical isomorphisms for $\mathcal{C} = \mathcal{H}, \mathcal{B}$

$$\mathsf{H}^{\mathsf{BM}}(\mathfrak{M}_{\mathcal{C}}, \mathbb{Q})_{\mathsf{vir}} \cong \mathsf{Sym}\left(\mathsf{BPS}_{\mathcal{C}} \otimes \mathsf{H}(\mathsf{B}\mathbb{C}^*, \mathbb{Q})\right)$$

Theorem

Combining these, there is a canonical NAHT isomorphism

$$\mathsf{H}^{\mathsf{BM}}(\mathfrak{M}_{\mathcal{H}}, \mathbb{Q})_{\mathsf{vir}} \cong \mathsf{H}^{\mathsf{BM}}(\mathfrak{M}_{\mathcal{B}}, \mathbb{Q})_{\mathsf{vir}}$$

By construction, the PS=WS conjecture is equivalent to the PI=WI conjecture, and (conjecturally) the P=W conjecture.