Differential calculus on double Lie Rinehart algebras

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Montpellier, 03-11-2020

Assume the basis is commutative

k is a field and unadorned tensor products are tensor products over k. Let A be a commutative k-algebra. The vector space of derivations of A, $Der(A)$, is endowed with a natural A-module structure.

Definition

A Lie Rinehart algebra (or Lie algebroid) over A is a triple $(L, [-,-], \omega)$ where

- $[-,-] : L \times L \to L$ is a k-Lie algebra
- L is an A-module
- $\omega : L \to Der(A)$ (the anchor) is an A-morphism and a Lie algebra morphism.
- $\forall X, Y \in L, \quad \forall a \in A$,

$$[X, aY] = \omega(X)(a)Y + a[X, Y].$$

Examples

Example 1: $L = Der(A)$ and $\omega = id$.
Example 2 : $A = k$. Then $\omega = 0$ and L is a k-Lie algebra.

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Example 3 :
Let $(N, \{-,-\})$ be a smooth Poisson manifold. Let $\pi \in \Gamma(N, \wedge^2(TN))$ be the bivector defining the Poisson bracket. Denote by $\pi^\sharp : \Omega^1(N) \to \Gamma(TN)$ the map defined by π. A Lie Rinehart algebra structure is defined on $\Omega^1(N)$ as follows :

$$[\alpha, \beta] = L_{\pi^\sharp \alpha}\beta - L_{\pi^\sharp \beta}\alpha - d\left(\pi(\alpha, \beta)\right).$$
$$\omega = \pi^\sharp$$

Example 4: (Huebschmann)
$(A, \{-,-\})$ is a Poisson algebra, the A-module of Kähler differentials D_A is endowed with a Lie Rinehart algebra structure over A as follows. For any $a, b, u, v \in A$,

- $[adu, bdv] = a\{u, b\}dv - b\{v, a\}du + ab\{u, v\}$
- $\omega(adu) = a\{u, -\}$.

There is a correspondance :

$$\{ \text{ Lie Rinehart algebra structures on } L \ \}$$
$$\updownarrow$$
$$\{ \text{ Gerstenhaber algebra structures on } \wedge L \}.$$

$\forall (D, \Delta) \in L, \quad \forall a \in A,$
$[D, a]_{\wedge_A L} = \omega(D)(a) \quad [D, \Delta]_{\wedge_A L} = [D, \Delta]_L$

On a Lie Rinehart algebra, there is a differential calculus generalizing that on a smooth manifold.

- A differential $d_L : Hom\,(\wedge^p L, A) \to Hom\,(\wedge^{p+1} L, A)$ defined by : If $\lambda \in Hom(\wedge^p L, A)$ and $X_1, \ldots, X_p \in L$,

$$d_L\lambda(X_1, \ldots, X_{n+1}) = \sum_i (-1)^{i-1}\omega(X_i)\left[\lambda\left(X_1, \ldots, \widehat{X}_i, \ldots, X_n\right)\right]$$
$$-\sum_{i<j}(-1)^{i+j}\lambda\left([X_i, X_j], X_1, \ldots, \widehat{X}_i, \ldots, \widehat{X}_j, \ldots X_n\right)$$

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  The differential is a degree one derivation.

- Contraction ι_D by an element D of L :

$$\iota_D\lambda(X_1, \ldots, X_{n-1}) = \lambda(D, X_1, \ldots, X_n)$$

  ι_D is a degree -1 derivation.

- Lie derivative by an element $D \in L$.

$$(L_D\lambda)(X_1, \ldots, X_n) = \\ \sum_i \lambda(X_1, \ldots, [D, X_i], \ldots X_n) - \omega(D)\,[\lambda(X_1, \ldots, X_n)]$$

L_D is a degree preserving derivation.

Examples

- If $L = Der(A)$, then d_L is the de Rham differential d_{dR}.
- if $A = k$, then L is a Lie algebra and d_L is the Cartan Eilenberg differential d_{CE}.
- If $(N, \{-,-\})$ is a Poisson smooth manifold, then $d_{\Omega^1(N)}$ is the Poisson -Lichnerovitch differential.

The algebra A is no longer supposed to be commutative.

We set $A^e = A \otimes A^{op}$.

The Kontsevich-Rosenberg philosophy

A theory over A in associative geometry has a meaning if it induces the classical notion over the representation spaces $Rep(A, n) = Hom_{alg}(A, Mat(n, k))$ for all $n \in \mathbb{N}$.
The algebra of regular functions over $Rep(A, n)$ is

$$\mathcal{O}_{Rep(A,n)} = \frac{k\left[a_{i,j}, \quad a \in A, \quad (i,j) \in [1,n]^2\right]}{< (ab)_{i,j} - \sum_k a_{i,k} b_{k,j} >}$$

The k-module $Der(A)$ has no structure, we will replace it by the double derivations :

$$\begin{aligned}\mathbb{D}er(A) &= \{D : A \to A \otimes A, \\ &\quad D(ab) = D(a)' \otimes D(a)'' b + aD(b)' \otimes D(b)''\}\end{aligned}$$

There are two A^e-module structures on $A \otimes A$:
- The outer one $\alpha \cdot (a \otimes b) \cdot \beta = \alpha a \otimes b\beta$
- The inner one $\alpha \cdot (a \otimes b) \cdot \beta = a\beta \otimes \alpha b$.

The exterior A^e-module structure is used in the definition of $\mathbb{D}er(A)$, the other one endows $\mathbb{D}$ with an A^e-module structure :

$$(\alpha D \beta)(a) = D(a)'\beta \otimes \alpha D(a)''.$$

To any double derivation D and any $(i,j) \in [1,n]^2$, we associate $D_{i,j} \in Der\left[\mathcal{O}_{Rep(A,n)}\right]$ defined by

$$D_{i,j}(a_{u,v}) = D(a)'_{u,j} D(a)''_{i,v}.$$

The trace map

$$\begin{array}{rcl} Tr : \mathbb{D}er(A) & \to & Der\left[\mathcal{O}_{Rep(A,n)}\right] \\ \mathbb{D} & \mapsto & \sum_i \mathbb{D}_{i,i}. \end{array}$$

Remark

The trace map does not see [-,-].

The Karoubi-de Rham complex

In associative geometry, the Karoubi-de Rham complex replaces the de Rham complex. Let A be an associative k-algebra not necessarily commutative.

Set $\Omega^1_A = Ker(m : A \otimes A \to A)$. It is an A^e-module. Put $da = a \otimes 1 - 1 \otimes a$. Then every element of Ω^1_A can be written $\sum a_i db_i$ and the relation $d(ab) = adb + (da)b$ is easy to check . Let us consider the complex $(\Omega^\bullet_A, d)$ defined as follows

$$\Omega^\bullet = T^\bullet_A(\Omega^1_A) = \oplus_n \Omega^n_A \quad \text{with} \quad \Omega^n_A = \Omega^1_A \otimes_A \cdots \otimes_A \Omega^1_A$$

and the differential d is defined by

$$\begin{array}{rcl} d : \Omega^n_A & \to & \Omega^{n+1}_A \\ adu_1 \otimes \cdots \otimes du_n & \mapsto & da \otimes du_1 \otimes \cdots \otimes du_n \end{array}$$

One can show easily that the complex $(\Omega^\bullet, d)$ is acyclic.

As the map d is a derivation, it induces an endomorphism of $\dfrac{\Omega_A^\bullet}{[\Omega_A^\bullet, \Omega_A^\bullet]}$ where $[\Omega_A^\bullet, \Omega_A^\bullet]$ denotes de k-linear space of all commutators.

Definition

The Karoubi - de Rham complex is defined by

$$DR_A := \left(\frac{\Omega_A^\bullet}{[\Omega_A^\bullet, \Omega_A^\bullet]}, d \right)$$

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Remark

If $\theta \in Der(A)$, the contraction by θ, ι_θ, and the Lie derivative L_θ were defined.

Double Gerstenhaber algebras

If $(V_i)_{i=1,\dots,n}$ are k graded vector spaces and $s \in S_n$, then for $a = a_1 \otimes \cdots \otimes a_n \in V_1 \otimes \cdots \otimes V_n$

$$\sigma_s(a) = (-1)^t a_{s^{-1}(1)} \otimes \cdots \otimes a_{s^{-1}(n)}.$$

where $t = \displaystyle\sum_{i<j, s^{-1}(i)>s^{-1}(j)} |a_{s^{-1}(i)}||a_{s^{-1}(j)}|.$

Definition

Let $d \in \mathbb{Z}$ and let D be a graded algebra. D is called a double Gerstenhaber algebra of degree d if it is equipped with a graded bilinear map

$$\{\{-,-\}\} : D \otimes D \to D \otimes D$$

of degree d such that the following identities hold :

1. $\{\{\alpha, \beta\gamma\}\} = (-1)^{(|\alpha|+d)|\beta|}\beta\{\{\alpha,\gamma\}\} + \{\{\alpha,\beta\}\}\gamma$
2. $\{\{\alpha,\beta\}\} = -(-1)^{(|\alpha|+d)(|\beta|+d)}\sigma_{(12)}\{\{\beta,\alpha\}\}$
3. $\{\{\alpha, \{\{\beta,\gamma\}\}\}\}_L$
   $+(-1)^{(|\alpha|+d)(|\beta|+|\gamma|)}\sigma_{(123)}\{\{\beta, \{\{\gamma,\alpha\}\}\}\}_L$
   $+(-1)^{(|\gamma|+d)(|\alpha|+|\beta|)}\sigma_{(132)}\{\{\gamma, \{\{\alpha,\beta\}\}\}\}_L = 0$

where

$$\{\{\{\{a,b\}\},c\}\}_L := \{\{\{\{a,b\}\}',c\}\} \otimes \{\{a,b\}\}''$$

If $(D, \{\{-,-\}\})$ is double Gerstenhaber algebra, then the bracket

$$\{a,b\} = \{\{a,b\}\}'\{\{a,b\}\}''$$

is a Leibniz bracket on D and $\dfrac{D}{[D,D]}[-d]$ is a graded Lie algebra. .

Remarks

The case $d = -1$ corresponds to double Gerstenhaber algebras (Van den Bergh) and the case $d = 0$ corresponds to double Poisson algebras (Van den Bergh). The extension to any d is due to Massuyeau and Turaev.

The double Lie Rinehart algebra $\mathbb{D}er(A)$

In this section, we recall results of Van den Bergh.

Let $Der_k(A, A \otimes A)$ be the space of derivations from A to $A \otimes A$ where we put the outer bimodule structure on $A \otimes A$. Such a derivation is called a double derivation. Inner bimodule structure on A^e allows to endow $Der(A, A \otimes A)$ with an A^e-module structure :

$$\alpha \cdot D \cdot \beta(a) = D(a)'\beta \otimes \alpha D(a)''$$

Notation : The A^e-module $Der(A, A \otimes A)$ will be denoted $\mathbb{D}er(A)$.

We will know put a double Gerstenhaber structure on $T_A(\mathbb{D}er(A))$.

- If $a \in A$ and $D \in \mathbb{D}er(A)$, $D(a) = \{\{D, a\}\} \in A \otimes A$ and it will be $D(a)' \otimes D(a)''$
- If D and Δ are in $\mathbb{D}er(A)$, then their bracket $\{\{D, \Delta\}\} \in \mathbb{D}er(A) \otimes A \ \oplus A \otimes \mathbb{D}er(A)$. We will write

$$\{\{D, \Delta\}\} = \{\{D, \Delta\}\}_l + \{\{D, \Delta\}\}_r$$

with

$$\{\{D, \Delta\}\}_l = \{\{D, \Delta\}\}'_l \otimes \{\{D, \Delta\}\}''_l \in \mathbb{D}er(A) \otimes A$$
$$\{\{D, \Delta\}\}_r = \{\{D, \Delta\}\}'_r \otimes \{\{D, \Delta\}\}''_r \in A \otimes \mathbb{D}er(A)$$

Proposition

(Van den Bergh) Let D, Δ in $\mathbb{D}er(A)$, then

$$\{\{D,\Delta\}\}_l^{\sim} = (D\otimes 1)\Delta - (1\otimes\Delta)D$$
$$\{\{D,\Delta\}\}_r^{\sim} := (1\otimes D)\Delta - (\Delta\otimes 1)D = -\{\{\Delta,D\}\}_l^{\sim}$$

are derivation from A to $A^{\otimes 3}$ with the outer A^e-module structure on $A^{\otimes 3}$.

We define

$$\{\{D,\Delta\}\}_l = \tau_{(23)}\circ\{\{D,\Delta\}\}_l^{\sim} \in \mathbb{D}er(A)\otimes A$$
$$\{\{D,\Delta\}\}_r = \tau_{(12)}\circ\{\{D,\Delta\}\}_r^{\sim} \in A\otimes\mathbb{D}er(A)$$

We write

$$\{\{D,\Delta\}\}_l = \{\{D,\Delta\}\}_l'\otimes\{\{D,\Delta\}\}_l''$$
$$\{\{D,\Delta\}\}_r = \{\{D,\Delta\}\}_r'\otimes\{\{D,\Delta\}\}_r''$$

with $\{\{D,\Delta\}\}_r', \{\{D,\Delta\}\}_l'' \in A$ and $\{\{D,\Delta\}\}_l', \{\{D,\Delta\}\}_r'' \in \mathbb{D}er(A)$.

We have $\{\{D,\Delta\}\}_r = -\tau_{12}\{\{\Delta, D\}\}_l$

Theorem

(Van den Bergh) For $a, b \in A$ and $D, \Delta \in \mathbb{D}er_B(A)$, the following equations

$$\begin{aligned}
&\{\{a,b\}\} = 0\\
&\{\{D,a\}\} = D(a) \in A \otimes A\\
&\{\{D,\Delta\}\} = \{\{D,\Delta\}\}_l + \{\{D,\Delta\}\}_r
\end{aligned}$$

define a unique structure of double Gerstenhaber algebra (of degree -1) on $T_A\mathbb{D}er(A)$.

Definition

(Van den Bergh) A double Lie Rinehart algebra $\mathbb{L}$ over A is an A^e-module together with a double Gerstenhaber structure (of degree -1) on $T_A(\mathbb{L})$.

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Examples

- $\mathbb{L} = \mathbb{D}er(A)$.
- If $A = k$,

$$\begin{array}{c}\text{Double Lie Rinehart structures over } \mathbb{L}\\ \updownarrow \\ \text{Associative algebra structures over } \mathbb{L}\end{array}$$

In this case $\{\{-,-\}\}_l$ is an associative product on A

- If $(A, \{\{-,-\}\})_P$ is a double Poisson algebra, then Ω^1_A is endowed with a double Lie algebra structure determined by

$$\{\{du, a\}\} = \{\{u, a\}\}$$
$$\{\{du, dv\}\} = d\{\{u, v\}\}' \otimes \{\{u, v\}\}'' + \{\{u, v\}\}' \otimes d\{\{u, v\}\}''$$

Before defining the differential of a double Lie Rinehart algebra, we will need to recall some algebra results and fix our conventions.

$A \otimes A$ is endowed with a $A^e \otimes A^e$-module structure. Let M be an A^e -module. We choose to set

$$M^* = \{\lambda : M \to A \otimes A \mid \lambda(\alpha \cdot D \cdot \beta) = \lambda(D)'\beta \otimes \alpha\lambda(D)''\}$$

M^* is itself an A^e-module as follows :

$$\forall \lambda \in M^*, \quad \forall (a, b) \in A^2, \quad \forall D \in M, \quad (a \cdot \lambda \cdot b)(D) = a\lambda(D)' \otimes \lambda(D)'' b.$$

Let M is an A^e- module. Let us endow $A^{\otimes n+1}$ with the $(A^e)^{\otimes n}$-module structure where the ith copy of A^e acts as follows :

$$a' \otimes a'' \cdot (a_1 \otimes \cdots \otimes a_{n+1}) = a_1 \otimes \cdots \otimes a_i a'' \otimes a' a_{i+1} \otimes \cdots \otimes a_{n+1},$$

Theorem

(Van den Bergh) The map

$$\Psi : M^{*\otimes_A n} \to Hom_{(A^e)^{\otimes n}}\left(M^{\otimes n}, A^{\otimes n+1}\right)$$

$$\Psi(\lambda_1 \otimes \lambda_2 \otimes \cdots \otimes \lambda_n)(m_1 \otimes \cdots \otimes m_n)$$
$$= \lambda_1(m_1)' \otimes \lambda_1(m_1)'' \lambda_2(m_2)' \otimes \cdots \otimes \lambda_{n-1}(m_{n-1})'' \lambda_n(m_n)' \otimes \lambda_n(m_n)''$$

is well defined.
If M is a finitely generated A^e-projective module, then Ψ is an isomorphism of A^e-modules.

Theorem

([SC]) Let $\mathbb{L}$ be a double Lie Rinehart algebra over A which is finitely generated projective as an A^e-module.
One defines $\underline{d}_{\mathbb{L}} : T_A(\mathbb{L}^*) \to T_A(\mathbb{L}^*)$ as the degree one derivation determined by :

- $\forall D \in \mathbb{L}, \quad \underline{d}_{\mathbb{L}} a(D) = D(a)' \otimes D(a)''$
- $\forall \lambda \in \mathbb{L}^*, \quad \underline{d}_{\mathbb{L}} \lambda \in \mathbb{L}^* \otimes_A \mathbb{L}^* \simeq Hom_{A^e}(\mathbb{L} \otimes \mathbb{L}, A^{\otimes 3})$ *is given by for all :* $D, \Delta \in \mathbb{L}$

$$\begin{aligned} \underline{d}_{\mathbb{L}}(\lambda)(D,\Delta) &= D\left(\lambda(\Delta)'\right) \otimes \lambda(\Delta)'' - \lambda(D)' \otimes \Delta\left(\lambda(D)''\right) \\ &- \lambda(\{\{D,\Delta\}\}_l')' \otimes \{\{D,\Delta\}\}_l'' \otimes \lambda(\{\{D,\Delta\}\}_l')'' \end{aligned}$$

One has $\underline{d}_{\mathbb{L}} \circ \underline{d}_{\mathbb{L}} = 0$.

Definition

The differential $\underline{d}_{\mathbb{L}}$ induces a differential

$$d_{\mathbb{L}} : \frac{T_A(\mathbb{L}^*)}{[T_A(\mathbb{L}^*), T_A(\mathbb{L}^*)]} \to \frac{T_A(\mathbb{L}^*)}{[T_A(\mathbb{L}^*), T_A(\mathbb{L}^*)]}.$$

We set

$$DR(\mathbb{L})^\bullet := \frac{T_A(\mathbb{L}^*)}{[T_A(\mathbb{L}^*), T_A(\mathbb{L}^*)]}$$

and $d_{\mathbb{L}}$ will be called "the differential of the double Lie algebroid" $\mathbb{L}$.

1. If $\mathbb{L} = \mathbb{D}er(A)$, we recover the Karoubi-de Rham complex but our formulas are different from the ones already existing.
2. If $A = k$, then $\{\{-,-\}\}_l$ endows $\mathbb{L}$ with an associative algebra and $DR(\mathbb{L})$ is the cyclic cohomology complex.
3. If $(A, \{\{-,-\}\})$ is a double Poisson algebra, then $\mathbb{L} = \Omega^1_A$ is a double Lie Rinehart algebra and $DR(\Omega^1_A)$ is the Poisson cohomology was defined by Pichereau-Van de Weyer.

To construct the contraction by a double derivation and the Lie derivative of a double derivation, we will make use of the following construction :
Let C be a graded algebra and let

$$\begin{array}{rcl} \Theta : C & \to & C \otimes C \\ c & \mapsto & \Theta(c)' \otimes \Theta(c)'' \end{array}$$

be a double derivation. One sets

$$\begin{array}{rcl} {}^\circ\Theta : C & \to & C \\ c & \mapsto & \Theta(c)''\Theta(c)'. \end{array}$$

${}^\circ\Theta$ is an endomorphism of C that cancels on $[C, C]$ and induces an endomorphism of $\dfrac{C}{[C, C]}$. We will be mostly interested in the case where $C = T_A(\mathbb{L}^*)$ or $C = T_A(\mathbb{L})$ where $\mathbb{L}$ is a double Lie–Rinehart algebra.

The contraction ([SC]

(similar to the construction of Crawley-Boevey- Etingof-Ginzburg)
Set $\mathbb{L}^* = Hom_{A^e}(\mathbb{L}, A \otimes A)$ and let $D \in \mathbb{L}$. The element D defines a degree -1 double derivation $\underline{\iota}_D : T_A(\mathbb{L}^*) \to T_A(\mathbb{L}^*) \otimes T_A(\mathbb{L}^*)$

$$\forall \alpha \in \mathbb{L}^*, \quad \underline{\iota}_D(\alpha) = \alpha(D)' \otimes \alpha(D)''.$$

It induces the restricted contraction $\iota_D =^\circ \underline{\iota}_D : DR(\mathbb{L}) \to DR(\mathbb{L})$.

Construction of the Lie derivative [SC]

The definition of the Lie derivative is more complicated for a general double Lie algebroid than in the case where $\mathbb{L} = \mathbb{D}er(A)$ ([CB-E-G]). Let D be an element of $\mathbb{L}$. The Lie derivative along D is the degree preserving derivation $\underline{L}_D : T_A(\mathbb{L}^*) \to T_A(\mathbb{L}^*) \otimes T_A(\mathbb{L}^*)$ defined by the following conditions :

- $\underline{L}_D(a) = D(a)$
- If $\lambda \in \mathbb{L}^*$,
  $\underline{L}_D(\lambda) \in A \otimes \mathbb{L}^* \oplus \mathbb{L}^* \otimes A$ is the sum of the maps

  $$\Delta \mapsto \lambda(\Delta)' \otimes D\left(\lambda(\Delta)''\right) - \tau_{(32)}\left[\lambda\left(\{\{D, \Delta\}\}_r''\right) \otimes \{\{D, \Delta\}\}_r'\right]$$

  in $\mathbb{L}^* \otimes A$ and the map

  $$\Delta \mapsto D\left(\lambda(\Delta)'\right) \otimes \lambda(\Delta)'' - \tau_{(32)}\left[\lambda(\{\{D, \Delta\}\}_l') \otimes \{\{D, \Delta\}\}_l''\right]$$

  in $A \otimes \mathbb{L}^*$

The double derivation $\underline{L}_\Theta$ induces the restricted Lie derivative $L_\Theta =^\circ (\underline{L}_\Theta)$.

Proposition

For any $\Theta \in \mathbb{L}$,

1. *The following equalities hold :*

$$\underline{d}_{\mathbb{L}} \circ \underline{\iota}_\Theta + \underline{\iota}_\Theta \circ \underline{d}_{\mathbb{L}} = \underline{\mathcal{L}}_\Theta, \quad \underline{d}_{\mathbb{L}} \circ \underline{\mathcal{L}}_\Theta = \underline{\mathcal{L}}_\Theta \circ \underline{d}_{\mathbb{L}}$$

2. *The maps $\underline{d}_{\mathbb{L}}$, $\underline{\mathcal{L}}_\Theta$ and $\underline{\iota}_\Theta$ descend to maps from $DR^\bullet(\mathbb{L})$ to $DR^\bullet(\mathbb{L})$ denoted respectively $d_{\mathbb{L}}$, $\mathcal{L}_\Theta$ and ι_Θ. One has*

$$d_{\mathbb{L}} \circ \iota_\Theta + \iota_\Theta \circ d_{\mathbb{L}} = \mathcal{L}_\Theta, \quad d_{\mathbb{L}} \circ \mathcal{L}_\Theta = \mathcal{L}_\Theta \circ d_{\mathbb{L}}$$

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2. *The maps $\underline{d}_{\mathbb{L}}$, $\underline{\mathcal{L}}_\Theta$ and $\underline{\iota}_\Theta$ descend to maps from $DR^\bullet(\mathbb{L})$ to $DR^\bullet(\mathbb{L})$ denoted respectively $d_{\mathbb{L}}$, $\mathcal{L}_\Theta$ and ι_Θ. One has*

$$d_{\mathbb{L}} \circ \iota_\Theta + \iota_\Theta \circ d_{\mathbb{L}} = \mathcal{L}_\Theta, \quad d_{\mathbb{L}} \circ \mathcal{L}_\Theta = \mathcal{L}_\Theta \circ d_{\mathbb{L}}$$

Remark

The previous proposition is proved for $\mathbb{L} = \mathbb{D}er(A)$ in CB-PE-VG but our proof is different.

THANK YOU FOR YOUR ATTENTION!