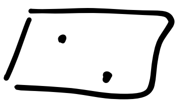
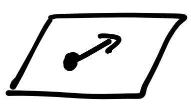
On the cohomology of Quot schemes of infinite affine space

Thanks, Toulouse!
work in progress with Joachim & Denis

Moduli spaces: useful & interesting

Examples: ① $\mathbb{P}^n$, $Gr_d(\mathbb{A}^n)$
smooth, stratification $\rightsquigarrow$ $H^*(Gr_d(\mathbb{A}^\infty_{\mathbb{C}}), \mathbb{Z}) \simeq \mathbb{Z}[c_1, \ldots, c_d]$

② Hilb: parametrize points (Grothendieck) configuration spaces where pts can collide

$Hilb_2(\mathbb{A}^n)$:  $\mathbb{A}^2$ &  $\mathbb{A}^2$

$$Hilb_d(X)(T) = \left\{ \begin{array}{c} Z \hookrightarrow X_T \\ {\scriptstyle p} \searrow \ \swarrow \\ T \end{array} \quad p \text{ finite } \overset{\text{(flat)}}{\text{loc-free}} \text{ of deg } d \right\},$$

$T = k \rightsquigarrow$ points, $Hilb = \coprod Hilb_d$, it's a scheme when X qproj.

Fun-facts:

- X smooth surface $\Rightarrow$ $Hilb(X)$ is smooth
- X K3 surface $\Rightarrow$ $Hilb(X)$ hyperkähler
- $Hilb(\mathbb{A}^n)$ satisfies Murphy's law when $n \geq 16$
- $Hilb(\mathbb{A}^n)$ is used for studying asymptotic bounds of matrix multiplication

③ Quot: quotients of sheaves

$Quot(X, \mathcal{F})$, $\mathcal{F} \in Coh(X)$, $\mathcal{F} = \mathcal{O}_X \rightsquigarrow$ get $Hilb(X)$.

Today: $X = \mathbb{A}^n$, $\mathcal{F} = \mathcal{O}^r_{\mathbb{A}^n}$. $S_n := \mathbb{Z}[x_1, \dots, x_n]$.

$\mathrm{Quot}_d(\mathbb{A}^n, \mathcal{O}^r)(\operatorname{Spec} R) = \{ S_{n,R}^{\oplus r} \twoheadrightarrow M$ - surjection of $S_n \otimes R$-mods s.t. M is loc-free rank d R-module$\}$

„choose r generators of M"

When $r=1$: quotients of $R[x_1, \dots, x_n]$ that are finite locally free over R, i.e. 0-dim subschemes of $\mathbb{A}^n_R$, which is exactly $\mathrm{Hilb}_d(\mathbb{A}^n)(\operatorname{Spec} R)$.

Goal: study cohomology / homotopy type of $\mathrm{Hilb}(\mathbb{A}^n)$ and $\mathrm{Quot}(\mathbb{A}^n, \mathcal{O}^r)$.

Let's start with Hilb.

$n=2$ $\mathrm{Hilb}(\mathbb{A}^2)$ is smooth, $\mathbb{G}_m \curvearrowright \mathbb{A}^2$

$\Rightarrow$ can use Białynicki-Birula decomposition to get a stratification by affine spaces from a $\mathbb{G}_m$-action.

Ex: $\mathbb{G}_m \curvearrowright \mathbb{P}^n$:

$$t(x_0, \dots, x_n) := (t x_0, t^2 x_1, \dots, t^{n+1} x_n).$$

fixed pts are $(0 : \dots : \overset{i}{1} : 0, \dots : 0)$ $\forall i$.

Under this action we have limits at 0 (continuous extension to $\mathbb{A}^1$), each limit is a fixed point, and strata (cells) are subsets with a fixed limit:

$$(0:\ldots:0:\underset{i}{1}:*:\ldots*) \simeq \mathbb{A}^{n-i},$$

which gives the usual cell structure on $\mathbb{P}^n$,

and $H^*(\mathbb{P}^n_{\mathbb{C}}, \mathbb{Z}) \simeq \mathbb{Z}[c_1]/c_1^{n+1}$ — generators correspond to strata

Problem: $n > 2 \Rightarrow$ Hilb is singular and so strata are not affine spaces anymore, so we can't compute H^*.

$n = \infty$: surprisingly easy answer via different methods!

Thm (HJNTY).

1) $H^*(\mathrm{Hilb}_d(\mathbb{A}^\infty_{\mathbb{C}}), \mathbb{Z}) \simeq \mathbb{Z}[c_1, \ldots, c_{d-1}]$, $|c_i| = 2i$

$$\mathbb{Z}[c_1, \ldots, c_{d-1}] \simeq H^*(\mathrm{Gr}_{d-1}(\mathbb{A}^\infty_{\mathbb{C}}), \mathbb{Z})$$

2) $H^*(\mathrm{Hilb}_d(\mathbb{A}^\infty_{\mathbb{C}}, \mathbb{Z})) \to H^*(\mathrm{Hilb}_d(\mathbb{A}^n_{\mathbb{C}}), \mathbb{Z})$

is an isom when $* \le 2n - 2d + 2$ (useful when $n \gg d$).

Rem. These computations work more generally for generalized coh. thys of smooth schemes A^*, by replacing $\mathbb{Z}$ with $A^*(S)$, S the base scheme: A^* can be ℓ-adic cohomology, KH, MGL etc.

Conj. (Rahul).

$$H^*(\mathrm{Quot}_d(\mathbb{A}^\infty_{\mathbb{C}}, \mathcal{O}^r), \mathbb{Z}) \cong \mathbb{Z}[c_1, \dots, c_d]/c_d^r$$

($r=1 \Rightarrow$ get previous Thm)

Main idea for Thm: $\mathrm{Gr}_{d-1}(\mathbb{A}^\infty) \longrightarrow \mathrm{Hilb}_d(\mathbb{A}^\infty)$

is an $\mathbb{A}^1$-htpy equivalence ($\approx \exists$ $\mathbb{A}^1$-family connecting them) $\Rightarrow$ values of $\mathbb{A}^1$-invariant coh. thys are the same.

Better to think of it at the level of stacks (more abstract & more explicit!):

$$\begin{tikzcd}
\mathrm{Gr}_{d-1}(\mathbb{A}^\infty) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{forgetful maps}"] & \mathrm{Hilb}_d(\mathbb{A}^\infty) \arrow[d] \\
\mathrm{Vect}_{d-1} \arrow[r] & \mathrm{FFlat}_d
\end{tikzcd}$$

proj. R-module $P \longmapsto R \oplus P$ square zero extension

One can write an $\mathbb{A}^1$-inverse map $A \mapsto A/_{R\cdot 1}$ with explicit $\mathbb{A}^1$-htpies composition $\rightsquigarrow$ id.

Problem: we don't know an analogue of the square zero extension map for Quot, so this argument doesn't work more generally.

Pre-Thm. (JNY). The forgetful map

$$\mathrm{Quot}_d(\mathbb{A}^n, \mathcal{O}^r) \longrightarrow \mathrm{Vect}_d^r := \{\text{rk. } d \text{ vector bdls with } r \text{ sections that don't vanish simultaneously}\}$$

is an $\mathbb{A}^1$-equivalence.

Rem. Pre-Thm $\Rightarrow$ Conj, because

$$H^*(\mathrm{Vect}_d^r, \mathbb{Z}) \simeq \mathbb{Z}[c_1, .., c_d]/c_d^r$$

$$(\mathrm{Vect}_d^r \hookrightarrow T_d^{\oplus r} \twoheadrightarrow \mathrm{Vect}_d).$$

Proof sketch.

Want: simplify the data of the points of Quot, stratify into pieces we understand and compare with $\mathrm{Vect}_d^{\sim}$.

1) $\mathrm{Quot} \underset{\text{open}}{\hookleftarrow} \mathrm{Quot}^{lin}$:

$$\begin{array}{ccc} S_{n,R}^{\oplus r} & \twoheadrightarrow & M \\ \cup & \nearrow & \\ (S_{n,R})_{\leq 1} & & \end{array}$$

linear part $-$ $(S_{n,R})_{\leq 1}$

It's an $\mathbb{A}^1$-htpy because the complement has codim $\xrightarrow[n\to\infty]{} \infty$: if $m \notin \mathrm{Im}(S_{n,R})_{\leq 1}$ you can add $(n+1)$-st variable that is sent to m

2) $M \in \mathrm{Quot}^{lin} \Rightarrow$ limit of M at 0 under torus action is some „square zero" module M':

$\Rightarrow$ there are few limits

$(S_{n,R}^{\oplus r})_{\geq 2} \cdot M' = 0$, so

$$M' = \underbrace{\mathrm{Im}(S_{n,R}^{\oplus r})_0}_{rk = k} \oplus \underbrace{\mathrm{Im}(S_{n,R}^{\oplus r})_1}_{rk = d-k}$$

with respect to limits

We can stratify Quot^{lin} into loc-closed strata $\mathrm{Quot}^{lin,k}$, $k \leq d$.

Under $\mathrm{Quot}_d(\mathbb{A}^\infty, \mathcal{O}^r) \to \mathrm{Vect}_d^{\sim}$

the strata $\mathrm{Quot}^{r,\mathrm{lin},k}$ corresponds to strata $\mathrm{Vect}_d^{r,k} \subset \mathrm{Vect}_d^r$,

$\{$image of r sections is k-dimensional$\}$

Claim: $\mathrm{Quot}_d^{\mathrm{lin},k} \to \mathrm{Vect}_d^{r,k}$ is an $\mathbb{A}^1$-equivalence (source is almost a vector bdl over target.)

It's easier than general case because $\mathrm{Quot}_d^{\mathrm{lin},k}$ has simpler data than Quot.

3) if $\mathrm{Quot}_d^r \to \mathrm{Vect}_d^r$ would be a map of smooth schemes, we could glue an $\mathbb{A}^1$-htpy (motivic) equivalence out of equivalences on pieces.

Quot_d^r is singular, but $\mathrm{Quot}_d^r \to \mathrm{Mod}_d$ is a smooth map and $\underset{S_n\text{-modules}}{\mathrm{Mod}_d} \xrightarrow[\mathbb{A}^1]{\sim} \mathrm{Vect}_d$, so we can work relatively over Mod and use 6 functor formalism to push down to the base.

So, singularities of Quot are transversal to our stratification, that's how we win!